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POTENTIAL VORTICITY MIXING IN HURRICANES: COMPARISON OF NONDIVERGENT AND DIVERGENT BAROTROPIC VORTICES

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1. INTRODUCTION

In a tropical cyclone, the concentrated potential vorticity (PV) source due to heating in a circular eyewall can result in a reversal of the radial PV gradient, allowing the vortex to become barotropically unstable. In this manner an axisymmetric vortex can develop asymmetries, redistribute PV through chaotic nonlinear mixing, and eventually resymmetrize with a different, stable structure. Recent studies of this process within the framework of an unforced nondivergent barotropic model (Schubert et al., 1998, herein referred to as S98) have helped provide insight into diverse aspects of hurricane dynamics, including the development of spiral bands and mesoscale vortices, the existence of polygonal eyewalls, and asymmetric eye contraction.

One approach to understanding the PV redistribution process is to treat it as an initial value problem. Numerical results from nonlinear models can simulate the process for a limited time but cannot track the details indefinitely, since the PV field tends to filament, developing structure on increasingly finer scales that eventually cannot be resolved. An alternative approach is to predict directly the final steady flow produced by the mixing, without simulating the time evolution of the flow. At least two approaches have been proposed, based on minimum enstrophy and maximum entropy arguments. This paper concentrates on the minimum enstrophy vortex (MinEV) problem, generalizing previous results for the unforced nondivergent barotropic model and extending the analysis to the divergent barotropic (shallow water) model.

2. NONDIVERGENT BAROTROPIC MODEL

In the unforced nondivergent barotropic model the enstrophy, energy, and angular momentum are all conserved in inviscid axisymmetric flow. With dissipation, none of the three are conserved, but the energy and angular momentum typically decay much more slowly than the enstrophy. Thus, at least for intermediate (non-diffusive) time scales, the flow may be thought to evolve toward a state of minimum enstrophy while conserving the energy and angular momentum. Starting from this "selective decay hypothesis" it is possible to derive the final state of the flow by variational techniques, yielding the MinEV solutions of Leith (1984) and S98. In this section we extend the approach of S98 to include two mixing radii and to constrain both energy and angular momentum in the same analysis.

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Consider an axisymmetric initial flow with tangential velocity $v_0(r)$ and corresponding vorticity $\zeta_0 = d(rv_0)/rdr$. If the flow is barotropically unstable, it may evolve through chaotic nonlinear mixing and asymptotically approach a steady final state which is again axisymmetric. We assume that the mixing is confined to the region $0 \le a < r < b$, and the flow outside this region is undisturbed. Denoting the tangential velocity and vorticity of the final state by $v(r)$ and $\zeta = d(rv)/rdr$, respectively, the enstrophy excess of the final state is

$$Z[v,a,b] := \int_a^b \tfrac{1}{2}\left(\zeta^2 - \zeta_0^2\right) rdr, \qquad (1)$$

where we have omitted the multiplicative constant 2π for convenience. The corresponding energy and angular momentum excesses are

$$E[v,a,b] := \int_a^b \tfrac{1}{2}\left(v^2 - v_0^2\right) rdr \qquad (2)$$

and

$$A[v,a,b] := \int_a^b r\,(v - v_0)\, rdr. \qquad (3)$$

We seek $v(r)$, a, and b which minimize $Z[v,a,b]$ subject to the constraints

$$E[v,a,b] = 0, \qquad A[v,a,b] = 0 \qquad (4)$$

and the boundary conditions

$$v(a) = v_0(a), \qquad v(b) = v_0(b). \qquad (5)$$

Using standard techniques from the calculus of variations, we find that the first variations of the boundary conditions (5) are

$$[\zeta(a) - \zeta_0(a)]\,\delta a + \delta v(a) = 0. \qquad (6)$$

and

$$[\zeta(b) - \zeta_0(b)]\,\delta b + \delta v(b) = 0. \qquad (7)$$

Using these, we can write the first variation of Z as

$$\delta Z = -\int_a^b \frac{d\zeta}{dr}\delta v\, rdr - \tfrac{1}{2}[\zeta(b) - \zeta_0(b)]^2\, b\delta b + \tfrac{1}{2}[\zeta(a) - \zeta_0(a)]^2\, a\delta a. \qquad (8)$$

The first variations of E and A are found in a similar manner to be

$$\delta E = \int_a^b v\delta v\, rdr \qquad (9)$$

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and

$$\delta A = \int_a^b r\delta v\, r dr, \tag{10}$$

where in both cases the boundary terms vanish due to (5).

To derive equations for the MinEV solution, we note that if v, a, and b minimize Z subject to the constraints of constant E and A, then they also minimize the functional

$$I[v,a,b] := Z[v,a,b] + \beta E[v,a,b] + \gamma A[v,a,b], \tag{11}$$

known as the Lagrangian, where β and γ are Lagrange multipliers. We can combine (8)–(10) with (11) to obtain

$$\begin{aligned}\delta I = &\int_a^b \left(-\frac{d\zeta}{dr} + \beta v + \gamma r\right) \delta v\, r dr \\ &- \tfrac{1}{2}\left[\zeta(b) - \zeta_0(b)\right]^2 b\delta b + \tfrac{1}{2}\left[\zeta(a) - \zeta_0(a)\right]^2 a\delta a.\end{aligned} \tag{12}$$

Since any stationary solution must have $\delta I = 0$ for all variations, we can obtain equations for the stationary solution as follows. First, consider variations for which $\delta a = 0$ and $\delta b = 0$ but δv is otherwise arbitrary. Then $\delta I = 0$ implies that the integrand in (12) must vanish, leading to the Euler-Lagrange equation

$$\frac{d}{dr}\left[\frac{d(rv)}{rdr}\right] - \beta v = \gamma r, \qquad a < r < b, \tag{13}$$

where we have substituted for ζ in terms of v. The solution of (13) subject to (5) may be obtained analytically in terms of Bessel functions and the parameters a, b, β, and γ. To determine these parameters, we substitute (13) into (12) and consider cases where $\delta a = 0$ and $\delta b = 0$ independently, concluding that the condition $\delta I = 0$ also implies the transversality conditions

$$\zeta(a) = \zeta_0(a), \qquad \zeta(b) = \zeta_0(b). \tag{14}$$

The conditions (14) and the constraints (4), i.e.,

$$\int_a^b \tfrac{1}{2}\left(v^2 - v_0^2\right) r dr = 0, \tag{15}$$

and

$$\int_a^b r\left(v - v_0\right) r dr = 0, \tag{16}$$

then give four conditions on a, b, β, and γ. The cases treated in S98 are recovered in the limit as $a \to 0$ with $\beta = 0$ (MinEV-M) and $\gamma = 0$ (MinEV-E).

To see whether the stationary solution defined by (13)–(16) in fact corresponds to minimum enstrophy (rather than a maximum or saddle point), we compute the second variation of I (at the stationary solution), obtaining

$$\delta^2 I = \tfrac{1}{2}\int_a^b \left\{\left[\frac{d(\delta v)}{dr}\right]^2 + \left(\beta + \frac{1}{r^2}\right)(\delta v)^2\right\} r dr. \tag{17}$$

From (17) we conclude that for nontrivial continuous variations δv satisfying $\delta v(a) = \delta v(b) = 0$ we have $\delta^2 I > 0$, provided that

$$\beta \geq -\frac{1}{b^2}. \tag{18}$$

Thus—if the condition (18) holds—the stationary solution defined by (13)–(14) and (4) corresponds to a minimum of I and thus is indeed the minimum enstrophy vortex.

3. DIVERGENT BAROTROPIC MODEL

The MinEV analysis for the axisymmetric divergent barotropic (shallow water) model on an f-plane parallels that developed above for the nondivergent model. Since the free surface height h is variable in this model, the key quantity conserved in inviscid flow is the potential vorticity

$$P = \frac{H}{h}\left[f + \frac{d(rv)}{rdr}\right], \tag{19}$$

where H is a constant reference height. We assume that the wind and mass fields are always in gradient balance, i.e.,

$$\left(f + \frac{v}{r}\right) v = g\frac{dh}{dr}, \tag{20}$$

where g is the acceleration due to gravity. Since the potential enstrophy, mass, total energy, and absolute angular momentum are all conserved in inviscid flow, the selective decay hypothesis suggests that the flow may evolve toward a state of minimum potential enstrophy while approximately conserving the mass, absolute angular momentum, and total energy. The problem is in one sense simpler than the nondivergent barotropic case, since the global dependence of the mass and wind field on the potential vorticity (through the invertibility principle) implies that there is no "mixing radius": the problem should be treated on the whole domain $0 \leq r < \infty$.

We can write the potential enstrophy, mass, total energy, and angular momentum in deviation form as

$$Z[h,v] := \int_0^\infty \tfrac{1}{2}(P-f)^2 h\, r dr, \tag{21}$$

$$M[h] := \int_0^\infty (h - H)\, r dr, \tag{22}$$

$$E[h,v] := \int_0^\infty \tfrac{1}{2}\left[v^2 h + g(h-H)^2\right] r dr, \tag{23}$$

and

$$A[h,v] := \int_0^\infty \left[rvh + \tfrac{1}{2} f r^2 (h - H)\right] r dr. \tag{24}$$

We assume that $P = f$ outside a bounded region (and that h is bounded); then it can be shown that v and $h - H$ both decay like $O(e^{-r})$ as $r \to \infty$, so each of Z, M, E, and A are finite. Thus, the MinEV problem is to find the flow field h, v in gradient balance which minimizes $Z[h,v]$ subject to the constraints $M[h] = M[h_0]$, $E[h,v] = E[h_0,v_0]$, and $A[h,v] = A[h_0,v_0]$, where h_0 and v_0 specify the initial flow (assumed to be axisymmetric and in gradient balance).

Using standard techniques from the calculus of variations we can compute the first variations of $Z[h,v]$, $M[h]$, $E[h,v]$, and $A[h,v]$ using (19) and (21)–(24), obtaining

$$\delta Z = \int_0^\infty \left[-\tfrac{1}{2}(P^2 - f^2)\delta h - H\frac{dP}{dr}\delta v\right] r dr, \tag{25}$$

$$\delta M = \int_0^\infty \delta h \, r dr, \tag{26}$$

$$\delta E = \int_0^\infty \left\{\left[\tfrac{1}{2}v^2 + g(h-H)\right]\delta h + vh\delta v\right\} r dr, \tag{27}$$

and

$$\delta A = \int_0^\infty \left[(rv + \tfrac{1}{2}fr^2)\delta h + rh\delta v\right] r dr. \tag{28}$$

To minimize Z while holding M, E, and A constant, we introduce the Lagrange multipliers α, β, and γ and seek to minimize the functional

$$I[h,v] := Z[h,v] + \alpha M[h] + \beta E[h,v] + \gamma A[h,v]. \tag{29}$$

Using (25)–(28) in (29) we obtain the first variation

$$\begin{aligned}\delta I = \int_0^\infty \Big[&-\tfrac{1}{2}(P^2 - f^2) + \alpha + \beta\left(\tfrac{1}{2}v^2 + g(h-H)\right) \\ &+ \gamma\left(rv + \tfrac{1}{2}fr^2\right)\Big]\delta h \, r dr \\ + \int_0^\infty &\left[-H\frac{dP}{dr} + \beta vh + \gamma rh\right]\delta v \, r dr,\end{aligned} \tag{30}$$

which must vanish if I is stationary at the solution h, v.

Before proceeding further, we note that δv and δh are not independent variations, since they are related by the constraint of gradient balance. We can construct a corresponding functional by multiplying (20) by an unspecified function $\mu(r)$ and integrating, obtaining

$$G[h,v] := \int_0^\infty \left[\left(f + \frac{v}{r}\right)v - g\frac{dh}{dr}\right]\mu \, r dr.$$

The corresponding first variation can be written as

$$\delta G = \int_0^\infty \left[\left(f + \frac{2v}{r}\right)\mu\delta v + g\frac{d(r\mu)}{rdr}\delta h\right] r dr. \tag{31}$$

Since the solution h, v is assumed to be in gradient balance we must have $\delta G = 0$, so we can add (31) to (30) to obtain

$$\begin{aligned}\delta I = \int_0^\infty \Big[&-\tfrac{1}{2}(P^2 - f^2) + \alpha + \beta\left(\tfrac{1}{2}v^2 + g(h-H)\right) \\ &+ \gamma\left(rv + \tfrac{1}{2}fr^2\right) + g\frac{d(r\mu)}{rdr}\Big]\delta h \, r dr \\ + \int_0^\infty &\left[-H\frac{dP}{dr} + \beta vh + \gamma rh + \left(f + \frac{2v}{r}\right)\mu\right]\delta v \, r dr.\end{aligned} \tag{32}$$

Since we must have $\delta I = 0$ for the stationary solution h, v, we can obtain Euler-Lagrange equations from (32) as follows. First, since the function $\mu(r)$ was arbitrary, we can choose it so that the term in brackets in the first integral in (32) vanishes, i.e.,

$$\begin{aligned} g\frac{d(r\mu)}{rdr} = &\tfrac{1}{2}(P^2 - f^2) - \alpha - \beta\left[\tfrac{1}{2}v^2 + g(h-H)\right] \\ &-\gamma\left(rv + \tfrac{1}{2}fr^2\right).\end{aligned} \tag{33}$$

Then the first integral in (32) vanishes, regardless of δh. Thus, the second integral must also vanish for all variations δv, so we conclude that the term in brackets in that integral must be zero, i.e.,

$$H\frac{dP}{dr} = \beta vh + \gamma rh + \left(f + \frac{2v}{r}\right)\mu, \tag{34}$$

These two equations, along with the definition of P

$$H\frac{d(rv)}{rdr} = hP - Hf, \tag{35}$$

and the gradient wind equation

$$g\frac{dh}{dr} = \left(f + \frac{v}{r}\right)v \tag{36}$$

form a system of ordinary differential equations to be solved for the four variables μ, P, v, and h. The boundary conditions on this system are

$$v(0) = 0, \qquad \mu(0) = 0 \tag{37}$$

(which come from examining the behavior of Taylor approximations to the solution as $r \to 0$) and

$$\lim_{r\to\infty} P(r) = f, \qquad \lim_{r\to\infty} h(r) = H. \tag{38}$$

Finally, the constraints on M, E, and A serve to determine the values of the Lagrange multipliers α, β, and γ in terms of the initial mass and velocity h_0 and v_0.

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